Improved Perturbation Theory for Improved Lattice Actions

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Abstract

We study a systematic improvement of perturbation theory for gauge fields on the lattice; the improvement entails resumming, to all orders in the coupling constant, a dominant subclass of tadpole diagrams.

This method, originally proposed for the Wilson gluon action [1], is extended here to encompass all possible gluon actions made of closed Wilson loops; any fermion action can be employed as well. The effect of resummation is to replace various parameters in the action (coupling constant, Symanzik coefficients, clover coefficient) by "dressed" values; the latter are solutions to certain coupled integral equations, which are easy to solve numerically.

Some positive features of this method are: a) It is gauge invariant, b) it can be systematically applied to improve (to all orders) results obtained at any given order in perturbation theory, c) it does indeed absorb in the dressed parameters the bulk of tadpole contributions.

Two different applications are presented: The additive renormalization of fermion masses, and the multiplicative renormalization Z_V (Z_A) of the vector (axial) current. In many cases where non-perturbative estimates of renormalization functions are also available for comparison, the agreement with improved perturbative results is significantly better as compared to results from bare perturbation theory.

Keywords: Lattice QCD, Perturbation theory, Improved actions, Tadpole improvement.

I. INTRODUCTION

Since the earliest studies of quantum field theories on a lattice, it was recognized that quantities measured through numerical simulation are characterized by significant renormalization effects, which must be properly taken into account before meaningful comparisons to corresponding physical observables can be made.

As has been rigorously demonstrated [2], the renormalization procedure can be formally carried out in a systematic way to any given order in perturbation theory. However, calculations are notoriously difficult, as compared to continuum regularization schemes; furthermore, the convergence rate of the resulting asymptotic series is often unsatisfactory.

A number of approaches have been pursued in order to improve the behaviour of perturbation theory, among them Refs. [3,4]. These approaches share in common the aim to reorganize perturbative series in terms of an expansion coefficient which would be more suitable than the bare coupling constant g_0; the definition of such a "renormalized" coupling constant is not unique, but can depend on the observables under study and on an energy scale. It is expected that such a definition will reabsorb a large part of the tadpole contibutions which are known to dominate lattice perturbation theory.

Some years ago, a method was proposed to sum up a whole subclass of tadpole diagrams, dubbed "cactus" diagrams, to all orders in perturbation theory [1,5]; this procedure has a number of desirable features: It is gauge invariant, it can be systematically applied to improve (to all orders) results obtained at any given order in perturbation theory, and it does indeed absorb the bulk of tadpole contributions into an intricate redefinition of the coupling constant; in cases where non-perturbative estimates of renormalization coefficients are also available for comparison, the agreement with cactus improved perturbative results is significantly better as compared to results from bare perturbation theory.

In the present work we extend the improved perturbation theory method of Refs. [1,5], to encompass the large class of actions which are used nowadays in simulations of QCD. This class includes Symanzik improved gluon actions with any arbitrary combination of closed Wilson loops, combined with any fermionic action. In Section II we present our calculation, deriving expressions for a dressed gluon propagator, as well as for dressed gluon and fermion vertices, as a result of the summation of cactus diagrams to all orders. We show how these dressed constituents are employed to improve 1-loop and 2-loop Feynman diagrams coming from bare perturbation theory. In Section III we apply our improved renormalizaton procedure to a number of test cases involving Symanzik gluons and Wilson/clover/overlap fermions. Improvement of QED is relegated to an Appendix.

Clearly, all resummation procedures, whether in the continuum or on the lattice, bear a caveat: A one-sided resummation could ruin desirable partial cancellations which might exist among those diagrams which are resummed and others which are not; what is worse, the end result might depend on the gauge. As we shall see, no partial cancellations will be ruined in our procedure, due to the distinct N-dependence of the resummed diagrams (N is the number of colors); furthermore, our results will be gauge independent.

II. THE METHOD

In this Section, following the outline of Ref. [1], we start illustrating our method by showing how the gluon propagator is dressed by the inclusion of cactus diagrams. We will then dress gluon and fermion vertices as well. Finally, we will explain how this procedure is applied to Feynman diagrams at a given order in bare perturbation theory, concentrating on the 1- and 2-loop case.

A. Dressing the propagator

We consider, for the sake of definiteness, the Symanzik improved gluon action involving Wilson loops with up to 6 links; it will be evident from what follows that the method is applicable to any gluon action made of Wilson loops. In standard notation (see, e.g., Ref. [6]), the action reads:

$$S_G = \frac{2}{g_0^2} \Bigg[c_0 \sum_{\text{plaquette}} \mathrm{Re\,Tr}\,(1 - U_{\text{plaquette}}) \, + \, c_1 \sum_{\text{rectangle}} \mathrm{Re\,Tr}\,(1 - U_{\text{rectangle}}) \\ + c_2 \sum_{\text{chair}} \mathrm{Re\,Tr}\,(1 - U_{\text{chair}}) \, + \, c_3 \sum_{\text{parallelogram}} \mathrm{Re\,Tr}\,(1 - U_{\text{parallelogram}}) \Bigg] \tag{1}$$

The coefficients c_i can in principle be chosen arbitrarily, subject to a normalization condition which ensures the correct classical continuum limit of the action:

$$c_0 + 8c_1 + 16c_2 + 8c_3 = 1 \tag{2}$$

Some popular choices of values for c_i used in numerical simulations will be considered in the applications of Section III. The quantities U_i ($i = 0, 1, 2, 3$, respectively: plaquette, rectangle, chair, parallelogram) in Eq. (1) are products of link variables $U_{x,\mu}$ around the perimeter of the closed loop.

Applying the usual parameterization of links in terms of the continuum gauge fields $A_\mu(x)$

$$U_{x,\mu} = \exp\Big(i\, g_0\, a\, A_\mu(x + a\hat{\mu}/2)\Big), \qquad A_\mu(x) = A_\mu^a(x)\, T^a, \quad \mathrm{Tr}\,(T^a\, T^b) = \tfrac{1}{2}\,\delta^{ab} \tag{3}$$

(a : lattice spacing, set to one from now on; $\hat{\mu}$: unit vector in direction μ ; T^a : generator of $SU(N)$ algebra) and the Baker-Campbell-Hausdorff (BCH) formula, U_i takes the form:

$$U_i = \exp\Big(i\, g_0\, F_i^{(1)} + i\, g_0^2\, F_i^{(2)} + i\, g_0^3\, F_i^{(3)} + \mathcal{O}(g_0^4)\Big) \tag{4}$$

where $F_i^{(1)}$ is simply the sum of the gauge fields on the links of loop i (e.g., for the plaquette: $F_0^{(1)} = A_\mu(x{+}\hat{\mu}/2) + A_\nu(x{+}\hat{\mu}{+}\hat{\nu}/2) - A_\mu(x{+}\hat{\nu}{+}\hat{\mu}/2) - A_\nu(x{+}\hat{\nu}/2))$, while $F_i^{(j)}$ ($j > 1$) are j-th degree polynomials in the gauge fields, constructed from nested commutators.

We may now *define* the cactus diagrams which dress the gluon propagator: These are tadpole diagrams which become disconnected if any one of their vertices is removed (see Fig. 1); further, each vertex is constructed solely from the $F_i^{(1)}$ parts of the action.

A diagrammatic equation for the dressed gluon propagator (thick line) in terms of the bare propagator (thin line) and 1-particle irreducible (1PI) vertices (solid circle) reads:

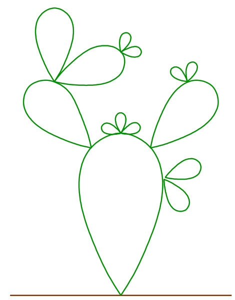
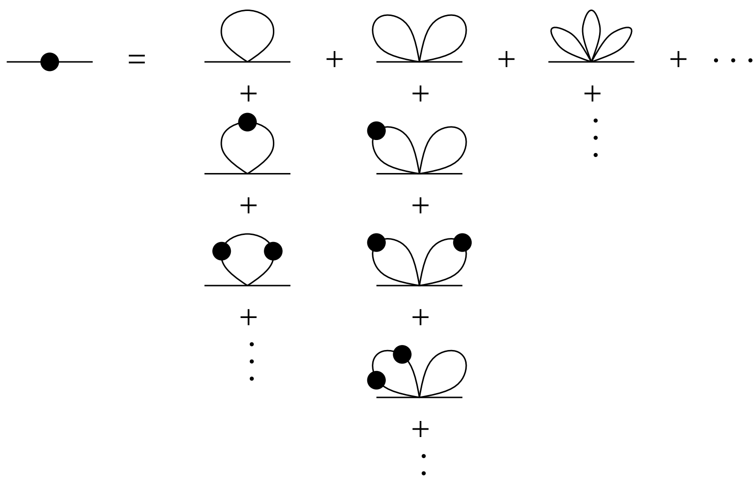


Figure 1: A cactus

 (5)

The 1PI vertex obeys the following recursive equation:

 (6)

In order to put this equation into a mathematical form and solve it, let us first write down the bare inverse propagator D^{-1} resulting from the action (1), and from the gauge fixing term:

$$S_{gf} = \frac{1}{1-\xi} \sum_{x,\mu,\nu} \mathrm{Tr}\left(\Delta_\mu^- A_\mu(x{+}\hat{\mu}/2)\, \Delta_\nu^- A_\nu(x{+}\hat{\nu}/2)\right), \qquad \Delta_\mu^- \phi(x) \equiv \phi(x{-}\hat{\mu}) - \phi(x) \quad (7)$$

The quadratic part of the total gluon action thus becomes (in the notation of Ref. [6]):

$$S_{\mathrm{G}}^{(0)} = \frac{1}{2} \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \sum_{\mu\nu} A_\mu^a(k)\, D_{\mu\nu}^{-1}(k)\, A_\nu^a(-k) \quad (8)$$

where: $$D_{\mu\nu}^{-1}(k) = \sum_\rho \left(\hat{k}_\rho^2 \delta_{\mu\nu} - \hat{k}_\mu \hat{k}_\rho \delta_{\rho\nu}\right) d_{\mu\rho} + \frac{1}{1-\xi}\, \hat{k}_\mu \hat{k}_\nu$$

and: $$d_{\mu\nu} = (1-\delta_{\mu\nu}) \left[C_0 - C_1\, \hat{k}^2 - C_2\, (\hat{k}_\mu^2 + \hat{k}_\nu^2)\right], \qquad \hat{k}_\mu = 2 \sin \frac{k_\mu}{2}, \quad \hat{k}^2 = \sum_\mu \hat{k}_\mu^2$$

The coefficients C_0, C_1, C_2 are related to the Symanzik coefficients c_i by

$$C_0 = c_0 + 8c_1 + 16c_2 + 8c_3\,,\ \ C_1 = c_2 + c_3\,,\ \ C_2 = c_1 - c_2 - c_3 \tag{9}$$

The inverse propagator can thus be put in the form:

$$D^{-1}_{\mu\nu}(k) \equiv c_0\, G^{(0)}_{\mu\nu}(k) + c_1\, G^{(1)}_{\mu\nu}(k) + c_2\, G^{(2)}_{\mu\nu}(k) + c_3\, G^{(3)}_{\mu\nu}(k) + \frac{1}{1-\xi}\,\hat{k}_\mu \hat{k}_\nu \tag{10}$$

The matrices $G^{(i)}(k)$ are symmetric and transverse, i.e. they satisfy:

$$\sum_\nu G^{(i)}_{\mu\nu}(k)\,\hat{k}_\nu = 0 \tag{11}$$

Each of them originates from a corresponding term: $\mathrm{Tr}(F^{(1)}_i\, F^{(1)}_i)$ of the gluon action. Consequently, each of the diagrams on the r.h.s. of Eq. (6), being the result of a contraction with only two powers of $F^{(1)}_i$ left uncontracted, will necessarily be equal to a linear combination of $G^{(i)}(k)$; this implies that the 1PI vertex $G^{\rm 1PI}(k)$ (the l.h.s. of Eq. (6)) can be written as:

$$G^{\rm 1PI}(k) = \alpha_0\, G^{(0)}(k) + \alpha_1\, G^{(1)}(k) + \alpha_2\, G^{(2)}(k) + \alpha_3\, G^{(3)}(k) \tag{12}$$

Each of the quantities α_i will in general depend on $N,\ g_0,\ c_0,\ c_1,\ c_2,\ c_3$, but not on the momentum. We must now turn Eq. (6) into a set of 4 recursive equations for α_i .

Eq. (5) leads to the following expression for the dressed propagator $D^{\rm dr}(k)$:

$$D^{\rm dr} = D + D\,G^{\rm 1PI}\,D + D\,G^{\rm 1PI}\,D\,G^{\rm 1PI}\,D + \cdots = D\left(\frac{1\!\!1}{1\!\!1 - G^{\rm 1PI}\,D}\right) \tag{13}$$

$$\begin{aligned} \Rightarrow\quad (D^{\rm dr})^{-1} &= (1\!\!1 - G^{\rm 1PI}\,D)\,D^{-1} = D^{-1} - G^{\rm 1PI} \\ &= \tilde{c}_0\, G^{(0)} + \tilde{c}_1\, G^{(1)} + \tilde{c}_2\, G^{(2)} + \tilde{c}_3\, G^{(3)} + \frac{1}{1-\xi}\,\hat{k}_\mu \hat{k}_\nu\,, \qquad \tilde{c}_i \equiv c_i - \alpha_i \end{aligned} \tag{14}$$

We observe that dressing affects entirely the transverse part of the inverse propagator, replacing the bare coefficients c_i with improved ones $\tilde{c}_i$, and leaves the longitudinal part intact. The same property carries over directly to the propagator itself; the consequence of this will be that our method leads to the *same results in all covariant gauges.*

In terms of the dressed propagator, Eq. (6) can be *drawn* as:

(15)

Let us now evaluate a typical diagram on the r.h.s. of Eq. (15); it will be the sum of 4 terms, one term for each of the Wilson loops U_i $(i = 0, 1, 2, 3)$ in the action, from which its n-point vertex may have originated. There are $(n-2)/2$ 1-loop integrals in the diagram; each of them corresponds to the contraction of two powers of $F^{(1)}_i$ via a dressed propagator, and will contribute one power of $\beta_i(\tilde{c}_0, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3)$, where:

$$\beta_0 = \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \left(2\,\hat{q}_\mu^2\, D^{\rm dr}_{\nu\nu}(q) - 2\,\hat{q}_\mu\,\hat{q}_\nu\, D^{\rm dr}_{\mu\nu}(q)\right)$$

$$\beta_1 = \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \Big((4\hat{q}_\nu^2 - \hat{q}_\nu^4)\, D^{\rm dr}_{\mu\mu}(q) + \hat{q}_\mu^2 (4{-}\hat{q}_\nu^2)\, D^{\rm dr}_{\nu\nu}(q) - 2\,\hat{q}_\mu\,\hat{q}_\nu (4{-}\hat{q}_\nu^2)\, D^{\rm dr}_{\mu\nu}(q) \Big)$$
$$\beta_2 = \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \Big(\hat{q}_\mu^2 (8{-}\hat{q}_\nu^2)\, D^{\rm dr}_{\rho\rho}(q)/2 - \hat{q}_\mu\,\hat{q}_\rho (8{-}\hat{q}_\nu^2)\, D^{\rm dr}_{\mu\rho}(q)/2 \Big)$$
$$\beta_3 = \int_{-\pi}^{\pi} \frac{d^4q}{(2\pi)^4} \Big(3\,\hat{q}_\mu^2 (4{-}\hat{q}_\nu^2)\, D^{\rm dr}_{\rho\rho}(q)/2 - 3\,\hat{q}_\mu\,\hat{q}_\nu\,(4{-}\hat{q}_\rho^2)\, D^{\rm dr}_{\mu\nu}(q)/2 \Big) \qquad (16)$$

(μ, ν, ρ assume distinct values; no summation implied). Once again, we note that β_i are gauge independent, since the longitudinal part cancels in the loop contraction.

For the contraction of the $SU(N)$ generators we first evaluate $F(n;N)$, which is the sum over all complete pairwise contractions of $\mathrm{Tr}\{T^{a_1}T^{a_2}\ldots T^{a_n}\}$:

$$F(2;N) = \delta_{a_1a_2}\,\mathrm{Tr}\,\{T^{a_1}\,T^{a_2}\}$$
$$F(4;N) = (\delta_{a_1a_2}\,\delta_{a_3a_4} + \delta_{a_1a_3}\,\delta_{a_2a_4} + \delta_{a_1a_4}\,\delta_{a_2a_3})\,\mathrm{Tr}\,\{T^{a_1}\,T^{a_2}\,T^{a_3}\,T^{a_4}\}$$
$$F(n;N) = \frac{1}{2^{n/2}(n/2)!} \sum_{P\in S_n} \delta_{a_1a_2}\,\delta_{a_3a_4}\ldots\delta_{a_{n-1}a_n}\,\mathrm{Tr}\left\{T^{P(a_1)}\,T^{P(a_2)}\ldots T^{P(a_n)}\right\} \qquad (17)$$

($F(2n+1;N) \equiv 0$; S_n is the permutation group of n objects). The generating function $G(z;N)$ for this quantity:

$$G(z;N) \equiv \sum_{n=0}^{\infty} \frac{z^n}{n!} F(n;N) \qquad \Rightarrow \qquad F(n;N) = \frac{d^n}{dz^n} G(z;N)\big|_{z=0} \qquad (18)$$

has been computed explicitly in Ref. [1], using Gaussian integration over the space of traceless Hermitian matrices [7], with the result:

$$G(z;N) = e^{z^2(N-1)/(4N)}\, L^1_{N-1}(-z^2/2) \qquad (19)$$

($L^\alpha_\beta(x)$: Laguerre polynomials). Since 2 out of n generators remain uncontracted in our case, color contraction does not lead to $F(n;N)$, but rather to:

$$\frac{n\,F(n;N)}{2(N^2{-}1)} \qquad (20)$$

Thus, upon contraction, an n-leg diagram in Eq. (15), with its vertex coming from the term U_i of the Langrangian ($i = 0,1,2,3$), will merely result in the following multiple of $G^{(i)}$:

$$\frac{c_i}{g_0^2}\,\frac{(i\,g_0)^n}{n!}\,\frac{n\,F(n;N)}{2(N^2{-}1)}\,4\,\beta_i^{(n-2)/2}\,G^{(i)} \qquad (21)$$

We are finally in a position to set Eq. (15) in a mathematical form:

$$\alpha_0\,G^{(0)} + \alpha_1\,G^{(1)} + \alpha_2\,G^{(2)} + \alpha_3\,G^{(3)} = \sum_{i=0}^{3}\ \sum_{n=4,6,8,\ldots}^{\infty} \frac{c_i}{g_0^2}\,\frac{(i\,g_0)^n}{n!}\,\frac{n\,F(n;N)}{2(N^2{-}1)}\,4\,\beta_i^{(n-2)/2}\,G^{(i)} \qquad (22)$$

Unknown in Eq. (22) are the coefficients α_i ; they appear on the l.h.s., as well as inside the integrals β_i of the r.h.s, by virtue of Eqs. (16, 14). We recall that $G^{(i)}$ are functions of the external momentum k ; if these are independent[1], then Eq. (22) amounts to 4 equations for the 4 coefficients α_i .

The generalization of our procedure for improved gluon actions with arbitrary numbers and types of Wilson loops is now evident. It is crucial to check at this stage that all combinatorial weights are correctly incorporated in Eq. (22); this is indeed the case.

Splitting Eq. (22) into 4 separate equations, and making use of Eq. (18), we can recast the infinite summations in closed form:

$$
\begin{aligned}
\frac{\alpha_i}{c_i} &= \sum_{n=4,6,8,\ldots}^{\infty} \frac{1}{g_0^2}\,\frac{(i\,g_0)^n}{n!}\,\frac{n\,F(n;N)}{2(N^2-1)}\,4\,\beta_i^{(n-2)/2} \\
&= 1 + \left(\sum_{n=0}^{\infty}\frac{(i\,g_0)^n}{n!}\,F(n{+}1;N)\,\beta_i^{n/2}\right)\frac{2(i\,g_0)}{g_0^2\,(N^2-1)}\,\beta_i^{-1/2} \\
&= 1 - \frac{2}{z\,(N^2-1)}\,G'(z;N)\Big|_{z=(i\,g_0\,\beta_i^{1/2})}
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\Rightarrow\qquad \frac{c_i-\alpha_i}{c_i}\,(N^2-1) &= \frac{2}{z}\,G'(z;N)\Big|_{z=(i\,g_0\,\beta_i^{1/2})} \\
&= e^{-\beta_i\,g_0^2\,(N-1)/(4N)}\,\left(\frac{N-1}{N}\,L^1_{N-1}(g_0^2\,\beta_i/2) + 2\,L^2_{N-2}(g_0^2\,\beta_i/2)\right)
\end{aligned}
\tag{24}
$$

In solving Eqs. (24), each choice of values for $(c_i\,,\,g_0\,,\,N)$ leads to a set of values for $\tilde{c}_i \equiv c_i - \alpha_i$. The latter are no longer normalized in the sense of Eq. (2); one may equivalently choose, however, to express the results of our procedure in terms of a normalized set of improved coefficients, $\tilde{c}_i/\tilde{C}_0$ and an improved coupling constant $\tilde{g}_0^2 = g_0^2/\tilde{C}_0$, where: $\tilde{C}_0 = \tilde{c}_0 + 8\tilde{c}_1 + 16\tilde{c}_2 + 8\tilde{c}_3$. In fact, it is convenient to treat bare and improved coefficients on an equal footing, by defining rescaled quantities as follows[2]:

$$
\gamma_i \equiv \frac{c_i}{g_0^2}\,, \qquad \tilde{\gamma}_i \equiv \frac{\tilde{c}_i}{g_0^2}\,, \qquad \tilde{\beta}_i(\tilde{c}_0,\tilde{c}_1,\tilde{c}_2,\tilde{c}_3) \equiv g_0^2\,\beta_i(\tilde{c}_0,\tilde{c}_1,\tilde{c}_2,\tilde{c}_3) = \beta_i(\tilde{\gamma}_0,\tilde{\gamma}_1,\tilde{\gamma}_2,\tilde{\gamma}_3) \tag{25}
$$

The rescaled quantities $\tilde{\gamma}_i$ must now satisfy the coupled equations:

$$
\tilde{\gamma}_i = \frac{1}{N^2-1}\,\gamma_i\,e^{-\tilde{\beta}_i\,(N-1)/(4N)}\,\left(\frac{N-1}{N}\,L^1_{N-1}(\tilde{\beta}_i/2) + 2\,L^2_{N-2}(\tilde{\beta}_i/2)\right) \tag{26}
$$

For the gauge groups $SU(2)$ and $SU(3)$, the Laguerre polynomials have a simple form, making Eqs. (26) more explicit:

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[1]Actually, $G^{(2)}$ is not independent of the rest, so that we have 3 equations for 3 coefficients; this causes no complication. In any case, typically $c_2 = 0$ in simulations.

[2]The dressed propagators in $\tilde{\beta}_i$ will now contain a rescaled gauge parameter $(1-\xi) \to g_0^2\,(1-\xi)$, which is irrelevant since the longitudinal part does not contribute.

$$(N=2)\ :\ \tilde{\gamma}_i = \gamma_i\, e^{-\tilde{\beta}_i/8}\left(1-\frac{\tilde{\beta}_i}{12}\right), \qquad (N=3)\ :\ \tilde{\gamma}_i = \gamma_i\, e^{-\tilde{\beta}_i/6}\left(1-\frac{\tilde{\beta}_i}{4}+\frac{\tilde{\beta}_i^2}{96}\right) \tag{27}$$

Given the highly nonlinear nature of Eqs. (26), it is not *a priori* clear that a solution for $\tilde{\gamma}_i$ always exists[3]; it turns out that this is always the case, for all physically interesting values of c_i, and for all values of g_0 ranging from $g_0 = 0$ up to a certain limit value, well inside the strong coupling region.

Fortunately, numerical solutions of Eqs. (26, 27) can be found very easily. We use a fixed point procedure, applicable to equations of the type $x = f(x)$:

$$\tilde{\gamma}_i = f_i(\tilde{\gamma}_i) \qquad \Rightarrow \qquad \tilde{\gamma}_i = \lim_{m\to\infty} \tilde{\gamma}_i^{(m)}, \qquad \text{where : } \tilde{\gamma}_i^{(0)} = \gamma_i\,, \quad \tilde{\gamma}_i^{(m+1)} = f_i(\tilde{\gamma}_i^{(m)}) \tag{28}$$

In order for the procedure to converge (attractive fixed point), it must be that: $|\partial f_i/\partial\tilde{\gamma}_i| < 1$ in a neighborhood of $\tilde{\gamma}_i$. This has been verified in a number of extreme cases.

B. Numerical values of improved coefficients

Here we present the values of the dressed coefficients for several gluon actions of interest.

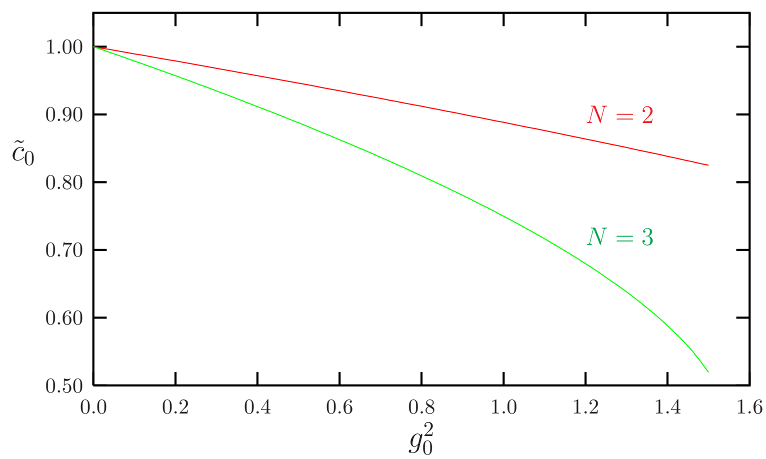


Fig. 2: Improved coefficient $\tilde{c}_0$ for N=2 and N=3 (plaquette action)

• Let us start with the plaquette action (c_0=1, c_1=c_2=c_3=0) [1]: In this case, Eqs. (26) reduce to only one equation, for $\tilde{\gamma}_0$, while $\tilde{\gamma}_i$=γ_i=0 (i=1, 2, 3). This equation is further simplified greatly since the integral $\tilde{\beta}_0$ can now be evaluated in closed form, $\tilde{\beta}_0 = 1/(2\tilde{\gamma}_0)$; for $N = 3$ we obtain (cf. Eq. (27)):

$$\tilde{c}_0 = e^{-g_0^2/(12\,\tilde{c}_0)}\left(1-\frac{g_0^2}{8\,\tilde{c}_0}+\frac{g_0^4}{384\,\tilde{c}_0^2}\right) \tag{29}$$

---

[3]The converse, of course, is trivial: Finding the bare values γ_i which lead to a given set of dressed values $\tilde{\gamma}_i$ is immediate, since the integrals $\tilde{\beta}_i$ only depend on $\tilde{\gamma}_i$, not γ_i .

In Fig. 2 we plot $\tilde{c}_0$ (in the notation of [1], $\tilde{c}_0 \equiv 1-w(g_0)$) as a function of g_0^2, for $N=2$ and $N=3$. The range of g_0 values, for which solutions exist, extends from $g_0^2=0$ (where $\tilde{c}_0=1$) up to $16\sqrt{e}/3 \simeq 3.23$ ($N=2$) and 1.558 ($N=3$); this covers the whole region of physical interest.

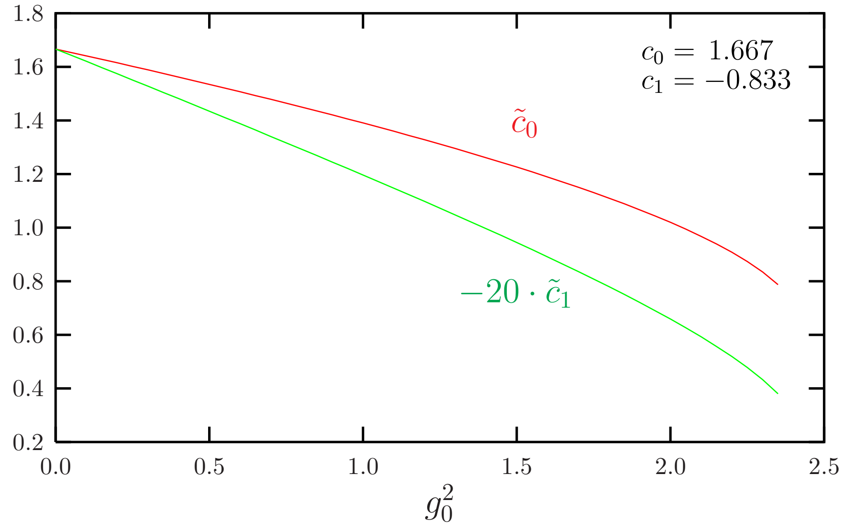


Fig. 3: Improved coefficients $\tilde{c}_0$ and $\tilde{c}_1$ (tree-level Symanzik improved action)

• The tree-level Symanzik improved action [8] corresponds to: c_0=5/3, c_1=$-1/12$, c_2=c_3=0. The dressed coefficients $\tilde{c}_0$, $\tilde{c}_1$ are shown in Fig. 3 for $N=3$.

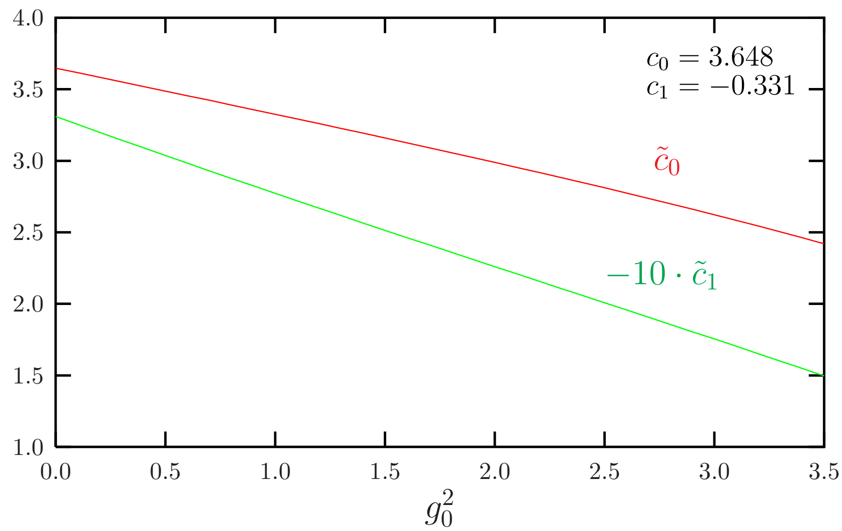


Fig. 4: Improved coefficients $\tilde{c}_0$ and $\tilde{c}_1$ (Iwasaki action)

• The Iwasaki set of parameter values [9] is: c_0=3.648, c_1=$-$ 0.331, c_2=c_3=0; while, in principle, c_0 and c_1 depend on g_0, they are typically kept constant in simulations. The corresponding dressed values are plotted in Fig. 4 ($N=3$).

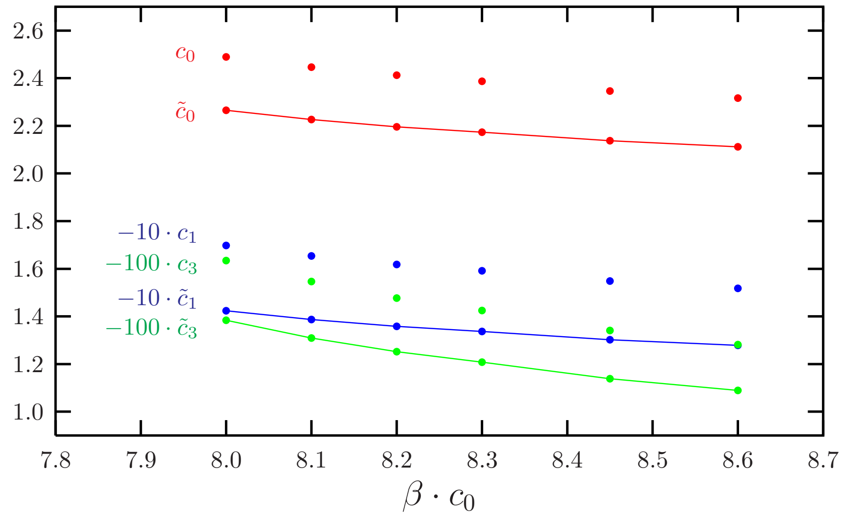


Fig. 5: Coefficients c_0 , c_1 , c_3 (red/blue/green dots, respectively) and their dressed counterparts $\tilde{c}_0$, $\tilde{c}_1$, $\tilde{c}_3$ (dots joined by a line), for different values of $\beta\, c_0 = 6\, c_0/g_0^2$ (TILW actions)

• Another class of gluon actions based on Symanzik improvement are the tadpole improved Lüscher-Weisz (TILW) actions [10,11]. In this case, the coefficients c_0 , c_1 , c_3 are optimized for each value of $\beta = 2N/g_0^2$ separately ($c_2 = 0$). In Fig. 5 we show the values of c_i and of their dressed counterparts $\tilde{c}_i$ in a typical range for β: $8.0 \le \beta\, c_0 \le 8.6$ ($N = 3$).

β	c_0	c_1	$\tilde{c}_0$	$\tilde{c}_1$
1.1636	5.29078	-0.53635	3.39826	-0.22528
0.6508	12.2688	-1.4086	8.8070	-0.7313

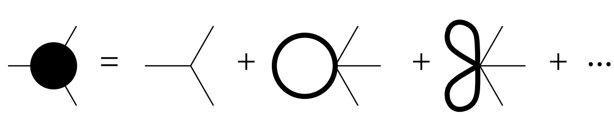
TABLE I. Improved coefficients $\tilde{c}_0$ and $\tilde{c}_1$ in the DBW2 action, for $\beta = 6/g_0^2 = 1.1636$ and 0.6508

• Finally, the DBW2 gluon action [12] corresponds to c_2=c_3=0, and β-dependent values for c_0, c_1. Some standard values for c_0 and c_1 (obtained *starting* from $\beta\, c_0 = 6.0$ and 6.3), as well as $\tilde{c}_0$ and $\tilde{c}_1$ are shown in Table I.

C. Dressing vertices

• We will begin by considering the 3-gluon vertex, coming from the action, Eq. (1). This vertex results from a Taylor expansion of U_i to 3rd order in g_0 . Expressing U_i as in Eq. (4), we see that only terms of the form $\mathrm{Tr}(F_i^{(1)}\, F_i^{(2)})$ will appear in this vertex, since $\mathrm{Tr}(F_i^{(3)})$ and $\mathrm{Tr}((F_i^{(1)})^3)$ will vanish.

By analogy with Eq. (15), the dressed 3-gluon vertex equals:



(30)

Consistently with the dressing of propagators, each $(2l+1)$-point vertex in Eq. (30) is a sum of 4 parts (one from each type of Wilson loop in the action), made up of

$$\mathrm{Tr}\big((F_i^{(1)})^{2l-1} F_i^{(2)}\big) \tag{31}$$

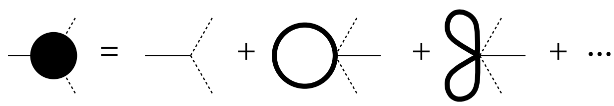
Denoting the bare 3-gluon vertex by: $V_3 = c_0\, V_3^{(0)} + c_1\, V_3^{(1)} + c_2\, V_3^{(2)} + c_3\, V_3^{(3)}$, it is relatively straightforward to see from Eq. (30) that the dressed vertex, $V_3^{\rm dr}$, is given by:

$$V_3^{\rm dr} = \sum_{i=0}^{3} c_i \left(\sum_{l=0}^{\infty} \frac{(i\, g_0)^{2l+1}}{(2l{+}1)!}\, \frac{2}{(N^2{-}1)}\, F(2l{+}2; N)\, \beta_i^l \right) (i\, g_0)^{-1}\, V_3^{(i)} \tag{32}$$

The summations inside parentheses are a mere multiple of those in Eq. (23); consequently, the result for $V_3^{\rm dr}$ turns out very simple:

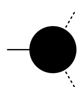
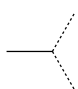
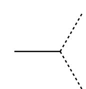
$$V_3^{\rm dr} = \tilde{c}_0\, V_3^{(0)} + \tilde{c}_1\, V_3^{(1)} + \tilde{c}_2\, V_3^{(2)} + \tilde{c}_3\, V_3^{(3)} \tag{33}$$

• We turn now to the 3-point fermion-antifermion-gluon vertex [5]. In the cases of Wilson and overlap fermions, these vertices remain unaffected, since the fermion actions do not contain any closed Wilson loops on which the BCH formula might be applied. The vertex from the clover action, on the other hand, is amenable to improvement; we write:



(34)

(fermions are denoted by a dotted line). Just as in Eq. (32), we find:

 $=$  $\cdot \left(\sum_{l=0}^{\infty} \frac{(i\, g_0)^{2l}}{(2l{+}1)!}\, \frac{2}{(N^2{-}1)}\, F(2l{+}2; N)\, \beta_0^l \right) =$  $\cdot \left(\frac{\tilde{c}_0}{c_0} \right)$ (35)

• Vertices with more fields would seem *a priori* more difficult to handle. To illustrate the complications that may arise, let us consider the 4-gluon vertex. The BCH expansion of $\mathrm{Tr}(U_i)$ contributes to this vertex in the form: $\mathrm{Tr}((F_i^{(1)})^4)$, $\mathrm{Tr}((F_i^{(2)})^2)$ and $\mathrm{Tr}(F_i^{(1)}\, F_i^{(3)})$. Such terms may in principle dress differently from each other. In addition, the dressed vertex produced from $\mathrm{Tr}((F_i^{(1)})^4)$ will not be a multiple of its bare counterpart; rather, it will be a linear combination of two color tensors (which are independent for $N > 3$):

$$\mathrm{Tr}\{T^a T^b T^c T^d + \text{permutations}\} \quad \text{and} \quad (\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}) \tag{36}$$

This issue has been resolved in Ref. [1], and it generalizes directly to the present case. Actually, such complications will not appear while dressing 1- and 2-loop diagrams in typical cases: Terms of the type $\mathrm{Tr}((F_i^{(1)})^4)$ must simply be omitted in order to avoid double counting, since their contribution is already included in dressing diagrams with one less loop. Thus, one is left only with: $\mathrm{Tr}((F_i^{(2)})^2)$ and $\mathrm{Tr}(F_i^{(1)}\, F_i^{(3)})$; for both of these terms it is straightforward to show, just as in Eqs. (32, 33), that their dressing amounts to replacing c_i by $\tilde{c}_i$.

• The same considerations as above apply to all higher vertices from both the gluon and fermion actions as well.

D. The improvement procedure in a nutshell

The steps involved in the resummation of cactus diagrams can now be described quite succinctly:

- Substitute gluon propagators in Feynman diagrams by their dressed counterparts. The latter are obtained by the replacement $c_i \to \tilde{c}_i = g_0^2\,\tilde{\gamma}_i$, where $\tilde{\gamma}_i$ are the solutions of Eqs. (26).
- Perform the same replacement, $c_i \to \tilde{c}_i$, on the 3-gluon vertex.
- Account for dressing of the 3-point vertex from the clover action by adjusting the clover coefficient $c_{\rm SW}: \; c_{\rm SW} \to c_{\rm SW} \cdot (\tilde{c}_0/c_0)$
- In dressing subleading-order diagrams, avoid double counting, i.e., subtract terms which were included in dressing leading-order diagrams. These are very easy to identify and subtract: Writing a general subleading-order result (aside from an overall prefactor) as: $a/N^2 + b + c\,N_f/N$ ($N_f:$ number of fermion flavors), the quantity to subtract will include all of a/N^2 (because terms with BCH commutators are higher order in N), and it will be a multiple of $(2N^2-3)$; thus subtraction boils down to the substitution:

$$a/N^2 + b + c\,N_f/N \to \left(b + \frac{2}{3}\,a\right) + c\,N_f/N \tag{37}$$

  (see Refs. [1,13–15] for different applications of this). The remaining subleading vertices dress exactly as the propagators and 3-point vertices above.

III. APPLICATIONS

We turn now to two different applications of cactus improvement: The additive mass renormalization for clover fermions and the 1-loop renormalization of the axial and vector currents using the overlap action. Both cases employ Symanzik improved gluons; hence, our results are presented for various sets of Symanzik coefficients.

A. Critical mass of clover fermions

It is well known that an ultra-local discretization of the fermion action without doubling breaks chirality. Consequently, we must demand a zero renormalized fermion mass, in order to ensure chiral symmetry while approaching the continuum limit. For this purpose, the bare mass is additively renormalized from its zero tree-level value to a critical value dm.

We compute dm using clover fermions and Symanzik improved gluons, in 1-loop perturbation theory; the result, denoted as $dm_{\rm 1-loop}$, is then dressed with cactus diagrams to arrive at the improved value $dm^{\rm dr}_{\rm 1-loop}$. The reader can refer to other works [13,14] for more details on the definition of dm. A two-loop calculation of dm with the same actions can be found in our Ref. [15]. There are only two 1-loop diagrams contributing to $dm_{1-\mathrm{loop}}$ and the result can be written as a polynomial in the clover parameter:

$$dm_{1-\mathrm{loop}} = \sum_{i=0}^{2} \varepsilon^{(i)} c_{\mathrm{SW}}^{i}$$
$$dm_{1-\mathrm{loop}}^{\mathrm{dr}} = \sum_{i=0}^{2} \varepsilon_{\mathrm{dr}}^{(i)} c_{\mathrm{SW}}^{i} \tag{38}$$

Clearly, one-loop results are independent of N_f, the number of fermion flavors. $\varepsilon_{\mathrm{dr}}^{(i)}$ includes one factor of $\tilde{c}_0/c_0$ for each power of c_{SW} (cf. Eq (35)). An overall factor of $g_0^2\,(N^2-1)/N$ has been absorbed in the coefficients $\varepsilon^{(i)}$ and $\varepsilon_{\mathrm{dr}}^{(i)}$, since the improvement procedure requires us to choose definite values of g_0 and N.

Some numerical values of Eqs. (38) corresponding to the plaquette and Iwasaki actions are given below ($N = 3$). First, for the plaquette action, choosing $\beta = 6.0$ one gets

$$\begin{aligned} dm_{1-\mathrm{loop}} &= -0.43428549(1) \;\; +0.1159547570(3)\; c_{\mathrm{SW}} \;\; +0.0482553833(1)\; c_{\mathrm{SW}}^2 \\ dm_{1-\mathrm{loop}}^{\mathrm{dr}} &= -0.579221119(2) \;\; +0.1159547570(3)\; c_{\mathrm{SW}} \;\; +0.0361806779(1)\; c_{\mathrm{SW}}^2 \end{aligned} \tag{39}$$

in agreement with Ref. [14]. For the Iwasaki action at $\beta = 1.95$:

$$\begin{aligned} dm_{1-\mathrm{loop}} &= -0.6773690760(3) \;\; +0.2342165224(9)\; c_{\mathrm{SW}} \;\; +0.0806966864(3)\; c_{\mathrm{SW}}^2 \\ dm_{1-\mathrm{loop}}^{\mathrm{dr}} &= -0.757856451(1) \;\; +0.1671007819(8)\; c_{\mathrm{SW}} \;\; +0.0447467282(1)\; c_{\mathrm{SW}}^2 \end{aligned} \tag{40}$$

Eqs. (39, 40) can be used to evaluate the critical hopping parameter κ_{cr} through:

$$\kappa_{\mathrm{cr}} \equiv \frac{1}{2\,dm + 8\,r} \tag{41}$$

where r is the Wilson parameter. Estimates of κ_{cr} from numerical simulations exist in the literature for the plaquette action [16,17] ($N_f = 0$), [18,19] ($N_f = 2$), and also the Iwasaki action [20] ($N_f = 2$). Perturbative (unimproved and dressed) and non-perturbative results are listed in Table II for specific values of c_{SW}. It is clear that cactus dressing leads to results for κ_{cr} which are much closer to values obtained from simulations.

Action	N_f	β	c_{SW}	$\kappa_{\mathrm{cr},1-\mathrm{loop}}$	$\kappa_{\mathrm{cr},1-\mathrm{loop}}^{\mathrm{dr}}$	$\kappa_{\mathrm{cr}}^{\mathrm{non-pert}}$
Plaquette	0	6.00	1.479	0.1301	0.1362	0.1392
Plaquette	0	6.00	1.769	0.1275	0.1337	0.1353
Plaquette	2	5.29	1.9192	0.1262	0.1353	0.1373
Iwasaki	2	1.95	1.53	0.1292	0.1388	0.1421

TABLE II. 1-loop results and non-perturbative values for κ_{cr}

B. One-loop renormalization of fermionic currents

As a second application of cactus improvement, we investigate the renormalization constant Z_V (Z_A) of the flavor non-singlet vector (axial) current in 1-loop perturbation theory. Overlap fermions and Symanzik improved gluons are employed. Bare 1-loop results for $Z_{V,A}$ have been computed in the literature [21,6,22]; they depend on the overlap parameter ρ ($0 < \rho < 2$).

One can show that, using the overlap action, the renormalization constants Z_V and Z_A are equal [21]; in the $\overline{MS}$ scheme they read

$$Z_{V,A}(a,p) = 1 - g_0^2\, z_{1V,A} \equiv 1 - g_0^2\, \frac{C_F}{16\pi^2}\,(b_{V,A} + b_\Sigma) \tag{42}$$

folowing the notation of [6,22]. $b_{V,A}$ and b_Σ are 1-loop results pertaining to the amputated two-point function of the current and to the fermion self-energy, respectively; since $Z_V = Z_A$, we can write $b_V = b_A$. Using cactus improvement, Eq. (42) becomes

$$Z^{\rm dr}_{V,A}(a,p) = 1 - g_0^2\ z^{\rm dr}_{1V,A} \tag{43}$$

To compute $z^{dr}_{1V,A}$ we dress the Symanzik coefficients and the propagators as described in the previous section. In Table III the values of $Z_{V,A}$ and $Z^{\rm dr}_{V,A}$ are presented for different sets of the Symanzik coefficients, choosing $\rho = 1.0$, $\rho = 1.4$. Systematic errors are too small to affect any of the digits appearing in the table. The dependence of $Z_{V,A}$ and $Z^{\rm dr}_{V,A}$ on the overlap parameter ρ is more clearly shown in Fig. 6, where we plot our results for three actions: Plaquette, Iwasaki and TILW. Note that improvement is more apparent for the case of the plaquette action. Indeed, from Table III one can clearly see that the effect of dressing is smaller for improved gluon actions. This, of course, could have been expected, since these actions were constructed in a way as to reduce lattice artifacts, in the first place.

Action	β=6/g_0^2	$Z_{V,A}(\rho$=1.0)	$Z^{\rm dr}_{V,A}(\rho$=1.0)	$Z_{V,A}(\rho$=1.4)	$Z^{\rm dr}_{V,A}(\rho$=1.4)
Plaquette	6.00	1.26427	1.35247	1.14707	1.19615
Symanzik	5.00	1.24502	1.29231	1.13574	1.16207
Symanzik	5.07	1.24164	1.28735	1.13386	1.15932
Symanzik	6.00	1.20418	1.23484	1.11311	1.13019
TILW	3.7120	1.27581	1.31941	1.15259	1.17690
TILW	3.6018	1.28223	1.32764	1.15613	1.18146
TILW	3.4772	1.28946	1.33677	1.16012	1.18651
TILW	3.3985	1.29434	1.34298	1.16282	1.18995
TILW	3.3107	1.29973	1.34972	1.16579	1.19369
TILW	3.2139	1.30569	1.35705	1.16908	1.19774
Iwasaki	1.95	1.39343	1.44921	1.21724	1.24847
Iwasaki	2.20	1.34872	1.38773	1.19256	1.21440
Iwasaki	2.60	1.29507	1.31940	1.16293	1.17656
DBW2	0.6508	1.49631	1.45362	1.27543	1.25057

TABLE III. Results for $Z_{V,A}, Z^{\rm dr}_{V,A}$ (Eq. (42,43)), using ρ=1.0, ρ=1.4

A comparison between our improved $Z_{V,A}$ values and some non-perturbative estimates [23], shows that improvement moves in the right direction. Cactus dressing had already been tested using clover fermions [5], and it turns out to be as good as standard tadpole improvement [4], but still not very close to non-perturbative results.

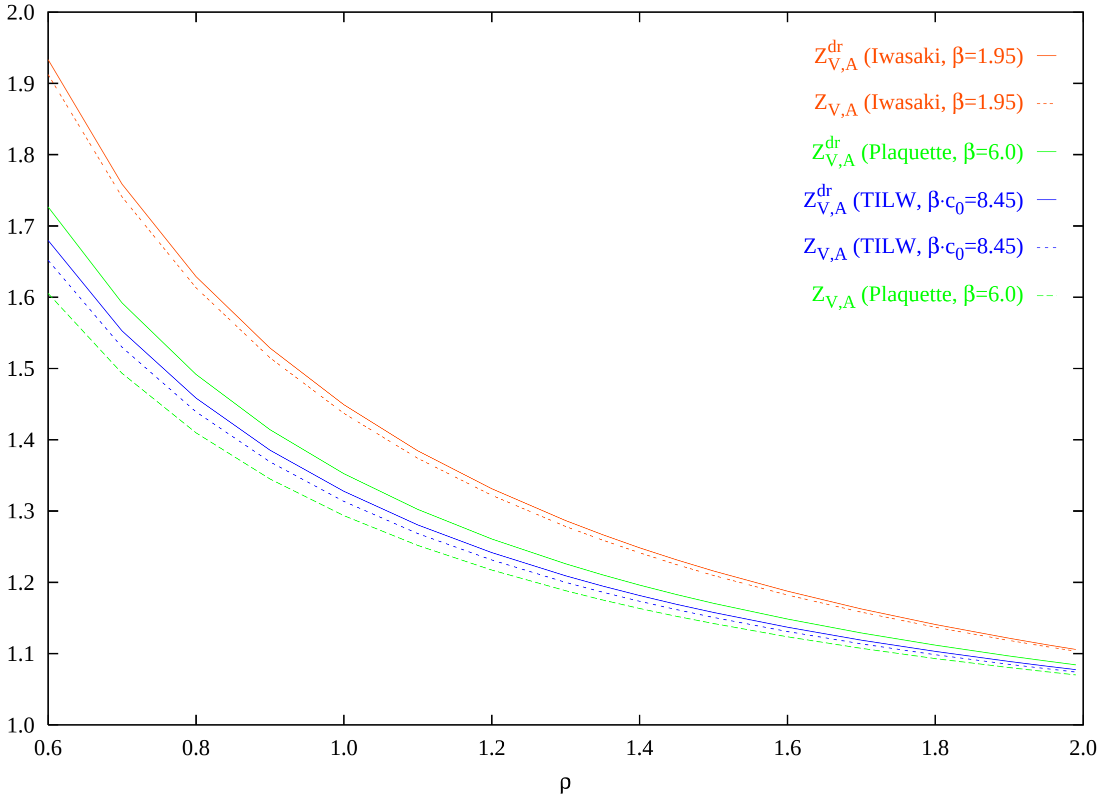


Fig. 6: Plots of $Z_{V,A}$ and $Z^{\rm dr}_{V,A}$ for the plaquette, Iwasaki and TILW actions. Labels have been placed in the same top-to-bottom order as their corresponding curves.

⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆

In closing, we remark that resummation of cactus diagrams is readily applicable to any observable in lattice gauge theories. This procedure for improving bare perturbation theory is gauge invariant, and can be applied in a systematic fashion to improve (to all orders) results obtained at any given order in perturbation theory.

APPENDIX A: DRESSING QED

Cactus improvement can be easily carried over to Lattice Quantum Electrodynamics. In this case, link variables commute; hence the first order BCH formula is exact and dressing includes the full contribution of diagrams with cactus topology.

Dressing the propagator now proceeds precisely as in Eq. (23). The only difference is that the result of contracting $n-2$ out of n generators of $SU(N)$: $n\,F(n;N)/(N^2-1)$, must now be replaced by: $\binom{n}{2}$ $(n-3)!!$ (the number of ways to pair $n-2$ out of n objects); this results in:

$$\begin{aligned}
\frac{\alpha_i}{c_i} &= \sum_{n=4,6,8,\ldots}^{\infty} \frac{1}{g_0^2}\,\frac{(i\,g_0)^n}{n!}\,\binom{n}{2}\,(n-3)!!\,2\,\beta_i^{(n-2)/2} = 1 - e^{-\beta_i\,g_0^2/2} \\
\Rightarrow \tilde{c}_i &\equiv c_i - \alpha_i = c_i\,e^{-\beta_i\,g_0^2/2}
\end{aligned} \tag{A1}$$

(β_i are the $\tilde{c}$-dependent integrals defined in Eqs. (16)).

As before, the 4 coupled equations (A1) for $\tilde{c}_i$ assume their simplest form in the case of the plaquette action ($c_0 = 1,\ c_1 = c_2 = c_3 = 0$); in this case, $\beta_0 = 1/(2\tilde{c}_0)$ and we obtain:

$$\tilde{c}_0 = e^{-g_0^2/(4\tilde{c}_0)}, \qquad \tilde{c}_1 = \tilde{c}_2 = \tilde{c}_3 = 0 \tag{A2}$$

Dressing vertices is simpler than in the non-Abelian case. For a bare $(2m)$-point vertex, denoted as: $V_{2m} = c_0\,V_{2m}^{(0)} + c_1\,V_{2m}^{(1)} + c_2\,V_{2m}^{(2)} + c_3\,V_{2m}^{(3)}$, instead of contracting group generators, one must simply count the number of distinct pairings of $2l$ objects out of $(2l{+}2m)$: $\binom{2l+2m}{2m}$ $(2l-1)!!$. The dressed vertex becomes:

$$\begin{aligned}
V_{2m}^{\rm dr} &= \sum_{i=0}^{3} V_{2m}^{(i)}\,c_i \sum_{l=0}^{\infty} \left(\frac{(i\,g_0)^{2l+2m}}{(2l{+}2m)!}\right)\left(\frac{(2m)!}{(i\,g_0)^{2m}}\right)\beta_i^l \binom{2l+2m}{2m}(2l-1)!! \\
&= \sum_{i=0}^{3} V_{2m}^{(i)}\,c_i\,e^{-\beta_i\,g_0^2/2} = \sum_{i=0}^{3} V_{2m}^{(i)}\,\tilde{c}_i
\end{aligned} \tag{A3}$$

The above relations allow us to summarize the dressing procedure for QED very briefly:

- Replace c_i by $\tilde{c}_i$ (as given in Eq. (A1)) throughout
- Omit all diagrams which contain any cactus subdiagram, to avoid double counting

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